

Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal

Diego Bombardelli¹, Davide Fioravanti¹ and Roberto Tateo²

¹ INFN-Bologna and Dipartimento di Fisica, Università di Bologna,
Via Irnerio 46, Bologna, Italy

² Dip. di Fisica Teorica and INFN, Università di Torino,
Via P. Giuria 1, 10125 Torino, Italy

e-mails:

bombardelli@bo.infn.it, fioravanti@bo.infn.it, tateo@to.infn.it

Abstract

Moving from the mirror theory Bethe-Yang equations proposed by Arutyunov and Frolov, we derive the thermodynamic Bethe Ansatz equations which should control the spectrum of the planar $\text{AdS}_5/\text{CFT}_4$ correspondence. The associated set of universal functional relations (Y-system) satisfied by the exponentials of the TBA pseudoenergies is deduced, confirming the structure inferred by Gromov, Kazakov and Vieira.

1 A bird's-eye view between integrability and AdS/CFT

A very peculiar phenomenon in modern theoretical physics has been taking place at the encounter of two branches: on one side the subject of quantum/statistical two-dimensional integrability [1] and on the other the gauge/string correspondences [2] in their planar case. Actually, the entrance of integrability into the realm of reggeised gluons of infinite colour QCD in its leading logarithmic approximation was already observed by Lipatov in [3].

More specifically, the AdS/CFT conjecture relates, by a strong/weak coupling duality, a type IIB superstring theory on the curved space-time $\text{AdS}_5 \times \text{S}^5$ and the conformal $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) in four dimensions on the boundary of AdS_5 [2]. As a consequence and particular case, the energy of a specific string state ought to be equal the anomalous dimension of the corresponding local gauge invariant operator in the quantum field theory. Yet, the mechanism of integrability in this triadic relation is not fully understood. For sure, the discovery of integrability in the classical string theory was a great achievement [4], both from the conceptual and the practical (i.e. calculative) point of view.

At the other side of the correspondence, in the maximally SYM theory for colour number $N \rightarrow \infty$ so that the 't Hooft coupling $N g_{YM}^2 = \lambda = 4\pi^2 g^2$, with g proportional to free string tension, is kept fixed, only the planar Feynman diagrams and single trace composite operators survive. Besides the pioneering interpretation of [5] in terms of a $sl(2)$ spin chain (in the QCD case), the constituent operators in the purely scalar sector at one loop have been unveiled to correspond to the degrees of freedom of an integrable $so(6)$ spin chain, thus making the mixing matrix (or dilatation operator) to coincide with this integrable $so(6)$ spin Hamiltonian [6]. Being integrable, the spectrum of this Hamiltonian comes out by means of the Bethe Ansatz (BA) (in one of its various forms) [1] and described by the so-called Bethe Ansatz equations for the 'rapidities' which parametrise the operators in the trace. Albeit a description of the dilatation operator at all loops as a spin chain Hamiltonian is still missing, the integrability has been showing up in the form of spin-chain-like Bethe equations (for g dependent rapidities still parametrising the operators in the trace, likewise to the one-loop case), which are valid at least in the *asymptotic regime* of large quantum numbers (cf. below). Eventually, a set of equations for the whole theory has been proposed by Beisert and Staudacher [7]. Computationally, the BA energy, $E(g)$, yields the anomalous part of the conformal dimension

$$\Delta = \Delta_{bare} + g^2 E(g), \quad (1.1)$$

where Δ_{bare} is the bare or classical dimension. As said before, this quantity must also be given by the quantum energy of a suitable string state ($E_{string} = \Delta$). By a semiclassical procedure on the string sigma model, this fact has opened a road to fix a phase factor, the so-called dressing factor, entering the Bethe equations (and the S -matrix) [8, 9, 10, 11]. Of course, Δ , Δ_{bare} and $E(g)$ may depend also on other quantum numbers, like the spin chain length L , – which also plays the rôle of a string angular momentum –, other angular momenta, the Lorentz spin s , etc.. Yet, the Beisert-Staudacher equations enjoy a validity seriously restricted by their scattering matrix origin, namely the length L and other quantum numbers need to be large. More precisely, starting from a certain loop order these equations are plagued by the so-called

‘wrapping problem’ [12, 13]. Nevertheless, as scattering S -matrix equations [14], they are indeed correct and they can be interpreted as Bethe-Yang quantisation conditions [15] [16].

In quantum integrable 2D relativistic massive field theories the problem of deriving off-shell quantities from on-shell information has been already addressed in many cases. For the purpose of this paper it is relevant the derivation by Al. B. Zamolodchikov of the finite-size ground state energy from the S -matrix [26]. Let us define the theory on a torus space-time geometry. The space direction is finite with circumference L , time is periodic with period $R \rightarrow \infty$. Zamolodchikov’s fascinating idea is to exchange space and time by defining a *mirror theory* in the infinite space R . In this mirror theory the space interval is infinite and the asymptotic Bethe-Yang equations hold true, but time is compact with size L . Now, we may interpret $L = 1/T$ as the inverse temperature and use the Yang-Yang thermodynamic Bethe Ansatz (TBA) procedure [23] to find the minimum free energy or equivalently the ground state energy for the (original) *direct* theory on a space circumference with size L . In the following, we will extend this procedure to the non-relativistic case relevant for the AdS/CFT correspondence.

We have been convinced that this strategy may be successful also in a complicated non-relativistic theory such as the AdS/CFT correspondence by the recent striking confirmation due to a sort of ancestor of the TBA for relativistic quantum field theory. In fact, Lüscher developed a method to compute, from scattering data, the finite-size corrections to the mass gap [17]. Later on, this method was specifically applied to integrable quantum field theories [18] and revealed itself as the leading term in the TBA large size expansion [29, 30]. Recently, a sophisticated extension of these ideas to the AdS/CFT correspondence has given striking results for the Konishi operator at four loops [19] and an impressive confirmation of the perturbative computations of [20].

In this article, we will start from the equations recently formulated by Arutyunov and Frolov in [21] for the mirror theory of the $\text{AdS}_5 \times S^5$ superstring theory. These equations are derived by implementing the classification of all the particles and bound states in the Bethe-Yang equations derived in [22]. The classification is obtained with the formulation of the so-called string hypothesis of the Hubbard model (cf. [24]): the map of the direct theory equations [15] into those of Hubbard’s was already observed by Beisert [16]. Initially, we will modify the equations – in analogy with those of the Hubbard model [24] –, so that we can take into account the information on the so-called $k - \Lambda$ strings. In this way, we produce a complete set of string equations for implementing the thermodynamic Bethe Ansatz method and derive a set of TBA equations for the single particle dressed energies (the pseudoenergies). As a conclusion, the pseudoenergies determine the (free) energy via a non-linear integral functional. We shall make explicit the similarity between our TBA equations and those for the Hubbard model and then derive a universal system of functional relations (the Y-system) for the exponential of the pseudoenergies. The universality of a Y-system consists in the fact that, at least for relativistic theories, it is the same for the excited states as well. Yet, there is by now a consolidated way towards excited states in relativistic massive field theories [29, 30, 31]. A very brief description of this procedure for the present case will be sketched in the final section, with the aim to gain a better control of the energy/dimension spectrum of the $\text{AdS}_5/\text{CFT}_4$ correspondence for any value of the coupling constant g and even for *short* operators. Apparently, the Y-system structure matches that recently proposed by Gromov, Kazakov and Vieira [37].

2 The equations for the root densities

As anticipated before, we need to pass from the $\text{AdS}_5 \times \text{S}^5$ theory defined on a circumference of length L to its mirror and this has been extensively investigated by Arutyunov and Frolov since the paper [22]. In particular, they derive from the S -matrix the Bethe-Yang equations for the fundamental particles of the mirror theory. Then, more recently [21], they extend these equations also to Q -particle bound states of the $\text{AdS}_5 \times \text{S}^5$ mirror theory in the form

$$\begin{aligned}
e^{i\tilde{p}_k R} &= \prod_{\substack{l=1 \\ l \neq k}}^{K^I} (S_0(\tilde{p}_k, \tilde{p}_l))^2 \prod_{\alpha=1}^2 \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{x_k^+ - y_l^{(\alpha)}}{x_k^- - y_l^{(\alpha)}} \sqrt{\frac{x_k^-}{x_k^+}}, \\
-1 &= \prod_{l=1}^{K^I} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_l^-}{x_l^+}} \prod_{l=1}^{K_{(\alpha)}^{III}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}, \\
1 &= \prod_{l=1}^{K_{(\alpha)}^{II}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}} \prod_{\substack{l=1 \\ l \neq k}}^{K_{(\alpha)}^{III}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}},
\end{aligned} \tag{2.1}$$

where

$$(S_0(\tilde{p}_k, \tilde{p}_l))^2 = \frac{x_k^- - x_l^+}{x_k^+ - x_l^-} \frac{1 - \frac{1}{x_k^+ x_l^-}}{1 - \frac{1}{x_k^- x_l^+}} \sigma^2(x_k, x_l) \tag{2.2}$$

is the $a = 0$ light-cone gauge scalar factor of the mirror S -matrix, with $\sigma(x_k, x_l)$ the dressing factor in the mirror theory [22]. Thanks to a so-far formal resemblance of the last two BA Equations (BAEs) with those of a inhomogeneous Hubbard model, they can formulate a string hypothesis for the solutions, in strict analogy with the Takahashi's one [24]. In few words, we assume that the thermodynamically relevant solutions ¹ of (2.1) in the limit of large $R, K^I, K_{(\alpha)}^{II}, K_{(\alpha)}^{III}$ rearrange themselves into complexes – the so-called strings – with real centers and all the other complex roots symmetrically distributed around these centers along the imaginary direction. Paying attention to the presence of two coupled Hubbard models for

¹There is no definitive proof of the string hypothesis, though it seems to give always the correct thermodynamic limit. There might well be other kinds of solutions (which should not affect the thermodynamics).

$\alpha = 1, 2$, the strings may be classified as follows:

- 1) N_Q Q -particles with real momenta \tilde{p}_k^Q and real rapidities u_k^Q :

$$u_k^{Q,j} = u_k^Q + (Q + 1 - 2j) \frac{i}{g}, \quad j = 1, \dots, Q; \quad (2.3)$$

- 2) $N_y^{(\alpha)}$ $y^{(\alpha)}$ -particles with real momenta $q_k^{(\alpha)}$;

- 3) $N_{M|v}^{(\alpha)}$ vw -strings with real centers v_k^M , $2M$ roots of type v and M of type w :

$$v_k^{M,j} = v_k^M \pm (M + 2 - 2j) \frac{i}{g}, \quad j = 1, \dots, M; \quad (2.4)$$

$$w_k^{M,j} = v_k^M + (M + 1 - 2j) \frac{i}{g}, \quad j = 1, \dots, M; \quad (2.5)$$

- 4) $N_{N|w}^{(\alpha)}$ w -strings with real centers w_k^N and N roots of type w :

$$w_k^{N,j} = w_k^N + (N + 1 - 2j) \frac{i}{g}, \quad j = 1, \dots, N. \quad (2.6)$$

If the variables u_k, v_k and w_k in (2.1) are replaced by $u_k^{Q,j}, v_k^{M,j}, w_k^{M,j}$ and $w_k^{N,j}$, and the products on the internal string index j are made, then the equations for the real centers of the various kinds of string (2.3) can be recast into the following form [21]

$$1 = e^{i\tilde{p}_k^Q R} \prod_{Q'=1}^{\infty} \prod_{\substack{l=1 \\ l \neq k}}^{N_{Q'}} S_{\mathfrak{sl}(2)}^{QQ'}(x_k, x_l) \prod_{\alpha=1}^2 \prod_{l=1}^{N_y^{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{vw}^{QM}(x_k, v_{l,M}^{(\alpha)}), \quad (2.7)$$

$$-1 = \prod_{Q=1}^{\infty} \prod_{l=1}^{N_Q} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_l^-}{x_l^+}} \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} \frac{v_k^{(\alpha)} - v_{l,M}^{(\alpha)-}}{v_k^{(\alpha)} - v_{l,M}^{(\alpha)+}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} \frac{v_k^{(\alpha)} - w_{l,N}^{(\alpha)-}}{v_k^{(\alpha)} - w_{l,N}^{(\alpha)+}}, \quad (2.8)$$

$$\prod_{Q=1}^{\infty} \prod_{l=1}^{N_Q} S_{vw}^{QK}(x_l, v_{k,K}^{(\alpha)}) = \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{vv}^{KM}(v_{k,K}^{(\alpha)}, v_{l,M}^{(\alpha)}) \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} S_{vw}^{KN}(v_{k,K}^{(\alpha)}, w_{l,N}^{(\alpha)}), \quad (2.9)$$

$$(-1)^K = \prod_{l=1}^{N_y^{(\alpha)}} \frac{w_{k,K}^{(\alpha)-} - v_l^{(\alpha)}}{w_{k,K}^{(\alpha)+} - v_l^{(\alpha)}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} S_{ww}^{KN}(w_{k,K}^{(\alpha)}, w_{l,N}^{(\alpha)}), \quad (2.10)$$

where, for shortness' sake, all the x -variables have to be read as

$$x_k^{\pm} \equiv x_k^{Q\pm} = x \left(u_k^Q \pm i \frac{Q}{g} \right), \quad (2.11)$$

and the definitions of the variables x^{\pm}, v, v_K^{\pm} and $w_K|\pm$ are reported in Appendix A. The S -matrices are defined as follows:

$$S_{\mathfrak{sl}(2)}^{QQ'}(x_k, x_l) = \left(\frac{x_k^{Q+} - x_l^{Q'-}}{x_k^{Q-} - x_l^{Q'+}} \right) \left(\frac{1 - \frac{1}{x_k^{Q-} x_l^{Q'+}}}{1 - \frac{1}{x_k^{Q+} x_l^{Q'-}}} \right) \sigma(x_k^{Q\pm}, x_l^{Q'\pm})^{-2}, \quad (2.12)$$

$$\begin{aligned} S_{xv}^{QM}(x_k, v_{l,M}) &= \left(\frac{x_k^{Q-} - x(v_{l,M}^+)}{x_k^{Q+} - x(v_{l,M}^+)} \right) \left(\frac{x_k^{Q-} - x(v_{l,M}^-)}{x_k^{Q+} - x(v_{l,M}^-)} \right) \left(\frac{x_k^{Q+}}{x_k^{Q-}} \right) \prod_{j=1}^{M-1} \left(\frac{u_k^Q - v_{l,M} - i \frac{Q-M+2j}{g}}{u_k^Q - v_{l,M} + i \frac{Q-M+2j}{g}} \right), \\ S_{vv}^{KM}(x, y) &= S_{vw}^{KM}(x, y) = S_{ww}^{KM}(x, y) = S_{KM}(x - y), \\ S_{KM}(u) &= \left(\frac{u + i \frac{|K-M|}{g}}{u - i \frac{|K-M|}{g}} \right) \left(\frac{u + i \frac{K+M}{g}}{u - i \frac{K+M}{g}} \right)^{\min(K,M)-1} \prod_{k=1}^{\min(K,M)-1} \left(\frac{u + i \frac{|K-M|+2k}{g}}{u - i \frac{|K-M|+2k}{g}} \right)^2, \end{aligned} \quad (2.13)$$

where $S_{\mathfrak{sl}(2)}^{QQ'}(x_k, x_l)$ is obtained from $(S_0(\tilde{p}_k, \tilde{p}_l))^2$ and the fusion procedure [45, 46]. Now, a simple crucial observation enters the stage: the last term in the r.h.s. of (2.9) fails the resemblance with the usual Hubbard BAEs implemented by string hypothesis [24, 25]. In fact, we need one more step: we can easily see that the equation for the vw strings – corresponding to the Hubbard $k - \Lambda$ strings – do not have in the r.h.s. a term of interaction between the w and vw strings; on the contrary there is a scattering term between a vw string and a single $v^{(\alpha)}$ (which do not belong to any string, but its own). Therefore, we may derive an intermediate equation

$$-1 = \prod_{l=1}^{N_y^{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} \frac{w_k^{(\alpha)} - w_{l,N}^{(\alpha)-} + \frac{i}{g}}{w_k^{(\alpha)} - w_{l,N}^{(\alpha)+} + \frac{i}{g}} \frac{w_k^{(\alpha)} - w_{l,N}^{(\alpha)-} - \frac{i}{g}}{w_k^{(\alpha)} - w_{l,N}^{(\alpha)+} - \frac{i}{g}}, \quad (2.14)$$

and choose $w_k^{(\alpha)}$ belonging to a vw -string. With this little trick ², we obtain

$$\prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} S_{vw}^{KN}(v_{k,K}^{(\alpha)}, w_{l,N}^{(\alpha)}) = (-1)^K \prod_{l=1}^{N_y^{(\alpha)}} \frac{v_{k,K}^{(\alpha)+} - v_l^{(\alpha)}}{v_{k,K}^{(\alpha)-} - v_l^{(\alpha)}}, \quad (2.15)$$

and finally we can rewrite (2.9) in a form re-echoing the Hubbard one

$$\prod_{Q=1}^{\infty} \prod_{l=1}^{N_Q} S_{xv}^{QK}(x_l, v_{k,K}^{(\alpha)}) = \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{vv}^{KM}(v_{k,K}^{(\alpha)}, v_{l,M}^{(\alpha)}) \prod_{N=1}^{\infty} \prod_{l=1}^{N_y^{(\alpha)}} S_{vy}^{KN}(v_{k,K}^{(\alpha)}, v_l^{(\alpha)}). \quad (2.16)$$

In (2.16) we have introduced a new scattering matrix

$$S_{vy}^{KN}(v_{k,K}^{(\alpha)}, v_l^{(\alpha)}) = \frac{v_{k,K}^{(\alpha)+} - v_l^{(\alpha)}}{v_{k,K}^{(\alpha)-} - v_l^{(\alpha)}} = \frac{v_{k,K}^{(\alpha)} - v_l^{(\alpha)} + iK/g}{v_{k,K}^{(\alpha)} - v_l^{(\alpha)} - iK/g}. \quad (2.17)$$

²After the first version of this paper appeared on the arXiv, this trick was implemented in a revised version of [21]

At this point, we can follow the standard TBA procedure [23, 24, 25, 26], which goes in a very sketchy way as follows. After taking the logarithm of these equations, we shall consider the thermodynamic limit $(K^I, N_y^{(\alpha)}, N_{vw}^{(\alpha)}, N_w^{(\alpha)}, R \rightarrow \infty)$ while keeping the densities finite (sums of root and hole densities, respectively)

$$\rho_Q(\tilde{p}) = \rho_Q^r(\tilde{p}) + \rho_Q^h(\tilde{p}) = \lim_{R \rightarrow \infty} \frac{I_{k+1}^Q - I_k^Q}{R(\tilde{p}_{k+1}^Q - \tilde{p}_k^Q)}, \quad (2.18)$$

$$\rho_y^\alpha(q) = \rho_y^{r\alpha}(q) + \rho_y^{h\alpha}(q) = \lim_{R \rightarrow \infty} \frac{I_{k+1}^{\prime\alpha} - I_k^{\prime\alpha}}{R(q_{k+1}^{(\alpha)} - q_k^{(\alpha)})}, \quad (2.19)$$

$$\rho_{v,K}^\alpha(\lambda) = \rho_{v,K}^{r\alpha}(\lambda) + \rho_{v,K}^{h\alpha}(\lambda) = \lim_{R \rightarrow \infty} \frac{J_{k+1}^{K\alpha} - J_k^{K\alpha}}{R(\lambda_{k+1}^{(\alpha)} - \lambda_k^{(\alpha)})}, \quad (2.20)$$

$$\rho_{w,K}^\alpha(\lambda) = \rho_{w,K}^{r\alpha}(\lambda) + \rho_{w,K}^{h\alpha}(\lambda) = \lim_{R \rightarrow \infty} \frac{J_{k+1}^{K\alpha} - J_k^{K\alpha}}{R(\lambda_{k+1}^{(\alpha)} - \lambda_k^{(\alpha)})}, \quad (2.21)$$

where the I s and the J s are the integer and half-integer quantum numbers. Eventually, we can produce for the thermodynamic state the following integral equations constraining the densities

$$\rho_Q(\tilde{p}) = \frac{1}{2\pi} + \sum_{Q'=1}^{\infty} (\phi_{\text{sl}(2)}^{QQ'} * \rho_{Q'}^r)(\tilde{p}) + \sum_{\alpha=1}^2 \left[(\phi_{xy}^Q * \rho_y^{r\alpha}) + \sum_{M=1}^{\infty} (\phi_{xv}^{QM} * \rho_{v,M}^{r\alpha}) \right] (\tilde{p}), \quad (2.22)$$

$$\rho_y^\alpha(q) = \sum_{Q=1}^{\infty} (\phi_{yx}^Q * \rho_Q^r)(q) + \sum_{M=1}^{\infty} (\phi_{yv}^M * \rho_{v,M}^{r\alpha})(q) + \sum_{N=1}^{\infty} (\phi_{yw}^N * \rho_{w,N}^{r\alpha})(q), \quad (2.23)$$

$$\rho_{v,K}^\alpha(\lambda) = \sum_{M=1}^{\infty} (\phi_{vv}^{KM} * \rho_{v,M}^{r\alpha})(\lambda) + (\phi_{vx}^{KQ} * \rho_Q^r)(\lambda) + (\phi_{vy}^K * \rho_y^{r\alpha})(\lambda), \quad (2.24)$$

$$\rho_{w,K}^\alpha(\lambda) = \sum_{M=1}^{\infty} (\phi_{ww}^{KM} * \rho_{w,M}^{r\alpha})(\lambda) + (\phi_{wy}^K * \rho_y^{r\alpha})(\lambda), \quad (2.25)$$

where the symbol $*$ denotes the *usual* convolution (on the second variable) $(\phi * g)(z) = \int dz' \phi(z, z') g(z')$ and the kernels are defined in Appendix A ³.

³We begin to notice that here the kernels $\phi(z, z')$ do not necessarily depend on the difference $(z - z')$.

3 The thermodynamic Bethe Ansatz equations

We continue our very sketchy presentation of the derivation of the TBA equations. For this purpose, we express the entropy in terms of the hole and root densities ⁴ (ρ^h and ρ^r , respectively)

$$\begin{aligned}
S = & \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} d\tilde{p} \left([\rho_Q^r(\tilde{p}) + \rho_Q^h(\tilde{p})] \ln [\rho_Q^r(\tilde{p}) + \rho_Q^h(\tilde{p})] - \rho_Q^r(\tilde{p}) \ln \rho_Q^r(\tilde{p}) - \rho_Q^h(\tilde{p}) \ln \rho_Q^h(\tilde{p}) \right) \\
& + \sum_{\alpha=1}^2 \int_{-\pi}^{\pi} dq \left([\rho_y^{r\alpha}(q) + \rho_y^{h\alpha}(q)] \ln [\rho_y^{r\alpha}(q) + \rho_y^{h\alpha}(q)] - \rho_y^{r\alpha}(q) \ln \rho_y^{r\alpha}(q) - \rho_y^{h\alpha}(q) \ln \rho_y^{h\alpha}(q) \right) \\
& + \sum_{\alpha=1}^2 \sum_{M=1}^{\infty} \int_{-\infty}^{\infty} d\lambda \left([\rho_{v,M}^{r\alpha}(\lambda) + \rho_{v,M}^{h\alpha}(\lambda)] \ln [\rho_{v,M}^{r\alpha}(\lambda) + \rho_{v,M}^{h\alpha}(\lambda)] - \rho_{v,M}^{r\alpha}(\lambda) \ln \rho_{v,M}^{r\alpha}(\lambda) \right. \\
& \quad \left. - \rho_{v,M}^{h\alpha}(\lambda) \ln \rho_{v,M}^{h\alpha}(\lambda) \right) \\
& + \sum_{\alpha=1}^2 \sum_{N=1}^{\infty} \int_{-\infty}^{\infty} d\lambda \left([\rho_{w,N}^{r\alpha}(\lambda) + \rho_{w,N}^{h\alpha}(\lambda)] \ln [\rho_{w,N}^{r\alpha}(\lambda) + \rho_{w,N}^{h\alpha}(\lambda)] - \rho_{w,N}^{r\alpha}(\lambda) \ln \rho_{w,N}^{r\alpha}(\lambda) \right. \\
& \quad \left. - \rho_{w,N}^{h\alpha}(\lambda) \ln \rho_{w,N}^{h\alpha}(\lambda) \right) , \tag{3.1}
\end{aligned}$$

and then minimise the free energy per unit length

$$f(T) = \tilde{H} - TS , \tag{3.2}$$

where \tilde{H} is the mirror energy per unit length [22]:

$$\tilde{H} = 2 \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} d\tilde{p} \operatorname{arcsinh} \left(\frac{\sqrt{Q^2 + \tilde{p}^2}}{2g} \right) \rho_Q^r(\tilde{p}) . \tag{3.3}$$

As stated before, then we ought to take as temperature T of the mirror theory the inverse of the size L in the AdS/CFT: $T = 1/L$. The extremum condition $\delta f = 0$ under the constraints (2.22)-(2.25) entails the final set of thermodynamic Bethe Ansatz equations for the pseudoenergies ϵ_A such that

$$\epsilon_A = \ln \frac{\rho_A^h}{\rho_A^r} , \quad \frac{1}{e^{\epsilon_A} + 1} = \frac{\rho_A^r}{\rho_A} , \quad L_A = \ln (1 + e^{-\epsilon_A}) , \tag{3.4}$$

⁴Hereafter the integration measure $d\tilde{p}$ has to be interpreted as Stieltjes measure $\frac{d\tilde{p}}{du} du$, as \tilde{p} depends on (the parameters) Q and g as well.

with the short indication of the collective index A for the different density labels. The ground state thermodynamic Bethe Ansatz equations are

$$\begin{aligned}\epsilon_Q(\tilde{p}) &= 2L \operatorname{arcsinh} \left(\frac{\sqrt{Q^2 - \tilde{p}^2}}{2g} \right) - \sum_{Q'=1}^{\infty} (\phi_{\mathfrak{sl}(2)}^{Q'Q} * L_{Q'}) (\tilde{p}) \\ &\quad - \sum_{\alpha=1}^2 (\phi_{yx}^Q * L_y^\alpha) (\tilde{p}) - \sum_{\alpha=1}^2 \sum_{M=1}^{\infty} (\phi_{vx}^{MQ} * L_{v,M}^\alpha) (\tilde{p}) ,\end{aligned}\tag{3.5}$$

$$\begin{aligned}\epsilon_y^\alpha(q) &= - \sum_{Q=1}^{\infty} (\phi_{xy}^Q * L_Q) (q) - \sum_{M=1}^{\infty} (\phi_{wy}^M * L_{w,M}^\alpha) (q) \\ &\quad - \sum_{N=1}^{\infty} (\phi_{vy}^N * L_{v,N}^\alpha) (q) ,\end{aligned}\tag{3.6}$$

$$\begin{aligned}\epsilon_{v,K}^\alpha(\lambda) &= - \sum_{Q=1}^{\infty} (\phi_{xv}^{QK} * L_Q) (\lambda) - (\phi_{yv}^K * L_y^\alpha) (\lambda) \\ &\quad - \sum_{M=1}^{\infty} (\phi_{vv}^{MK} * L_{v,M}^\alpha) (\lambda) ,\end{aligned}\tag{3.7}$$

$$\epsilon_{w,K}^\alpha(\lambda) = - (\phi_{yw}^K * L_y^\alpha) (\lambda) - \sum_{M=1}^{\infty} (\phi_{ww}^{MK} * L_{w,M}^\alpha) (\lambda) ,\tag{3.8}$$

with $\alpha = 1, 2$, $Q = 1, 2, \dots$ and $K = 1, 2, \dots$. Notice that, apart from the specific form of the kernels (see Appendix A for their definitions), the TBA equations are similar in form to the density equations (2.22-2.25), provided we exchange $\rho \rightarrow -L$. However, we should stress that on our way from (2.22-2.25) to (3.5-3.8) we have made an abuse of notation and changed definition for the convolution $*$ moving on to the first variable

$$(\phi * g)(z) = \int dz' \phi(z', z) g(z') .\tag{3.9}$$

When the kernel $\phi(z, z')$ depends only on the difference $|z - z'|$, as for example in the relativistic theories discussed in [26, 28], this change in the definition of $*$ can be avoided by keeping the convolution on the second variable. However, in the present framework some of the kernels have a genuinely different functional dependence on the two independent variables and this simplification is absent. Moreover, an important comment is here on the integration limits: they are from $-\infty$ to ∞ for the λ - and \tilde{p} -variables but from $-\pi$ to π for the q -variables.

As a concluding result, the *minimal* free energy for the mirror theory results by inserting the TBA equations into the general (3.2) and is given by the following non-linear functional of the pseudoenergies $\epsilon_Q(u)$

$$f(T) = -T \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\tilde{p}}{2\pi} \ln(1 + e^{-\epsilon_Q(\tilde{p})}) = -T \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{d\tilde{p}}{du} \ln(1 + e^{-\epsilon_Q(u)}) .\tag{3.10}$$

Consequently, the ground state energy for the AdS/CFT theory on a circumference with length $L = 1/T$ ought to satisfy the relation

$$E_0(L) = Lf(1/L) . \quad (3.11)$$

As we have kept the total densities finite, it is natural to introduce chemical potentials μ_A . This has been already finalised in relativistic theories by [28]. The TBA equations (3.5–3.8) do not change their form, but for this simple replacement

$$L_A = \ln(1 + e^{-\epsilon_A}) \rightarrow L_{A,\lambda} = \ln(1 + \lambda_A e^{-\epsilon_A}) , \quad (3.12)$$

involving the fugacities $\lambda_A = e^{\mu_A/T}$. Here, we would like to conjecture that their introduction should be related to the zero energy of the ground state (independently of the value of T) which is a half BPS protected state. It is a consequence of a result by [33], further developed in [32] and in [34] that in particular $\mathcal{N} = 2$ supersymmetric theories this size invariant state can be selected via a suitable tuning of the TBA fugacities. A plot describing this interesting transition, as the fugacities approach these critical values can be found in [35]. In our case we expect zero energy as soon as the fugacities reach these values

$$\lambda_Q = 1 , \quad \lambda_{v,K}^\alpha = -1 , \quad \lambda_{w,K}^\alpha = (-1)^{K+1} , \quad \lambda_y^\alpha = -1 , \quad (3.13)$$

($\alpha = 1, 2, K = 1, 2, \dots$).

Physically, this modification corresponds to the calculation of the Witten index. In (3.13), the fermionic and bosonic character of the pseudoparticles is chosen following an analogy with other scattering-matrix models and considering the evident Z_2 -symmetry of the TBA equations. There are -of course- other possibilities. The vanishing of ground state energy in TBA models is a very delicate issue and we prefer to postpone this discussion to the near future and in presence of analytic or numerical evidences.

3.1 A comparison with the Hubbard TBA equations

As the reader can see in Appendix A, some kernels in (3.5)–(3.8) actually depends on the difference of rapidities. Therefore, the convolutions involving these kernels is a standard ‘difference’ convolution, i.e. $(f * g)(z) = \int dz' f(z - z') g(z')$. In other words, we may rewrite the equations (3.6)–(3.8) in a form that is closer to the TBA equations of the Hubbard model, as we might expect from the analogy at the level of Bethe Ansatz equations. Of course, we must leave untouched the terms really depending on the two different variables and think of them as *driving* or *forcing* terms connecting the two Hubbard models. For this reason, we move them on the

l.h.s. of the equations and write

$$\begin{aligned} \epsilon_y^\alpha(q) + \sum_{Q=1}^{\infty} (\phi_{xy}^Q * L_Q)(q) &= \sum_{M=1}^{\infty} \int_{-\infty}^{\infty} d\lambda \, a_M(\lambda - \sin(q)) \ln(1 + e^{-\epsilon_{v,M}^\alpha(\lambda)}) \\ &\quad - \sum_{M=1}^{\infty} \int_{-\infty}^{\infty} d\lambda \, a_M(\lambda - \sin(q)) \ln(1 + e^{-\epsilon_{w,M}^\alpha(\lambda)}) , \end{aligned} \quad (3.14)$$

$$\begin{aligned} \epsilon_{v,K}^\alpha(\lambda) + \sum_{Q=1}^{\infty} (\phi_{xv}^{QK} * L_Q)(\lambda) &= - \int_{-\pi}^{\pi} dq \, \cos(q) a_K(\sin(q) - \lambda) \ln(1 + e^{-\epsilon_y^\alpha(q)}) \\ &\quad + \sum_{M=1}^{\infty} (A_{MK} * L_{v,M}^\alpha)(\lambda) , \end{aligned} \quad (3.15)$$

$$\begin{aligned} \epsilon_{w,K}^\alpha(\lambda) &= - \int_{-\pi}^{\pi} dq \, \cos(q) a_K(\sin(q) - \lambda) \ln(1 + e^{-\epsilon_y^\alpha(q)}) \\ &\quad + \sum_{M=1}^{\infty} (A_{MK} * L_{w,M}^\alpha)(\lambda) , \end{aligned} \quad (3.16)$$

where

$$a_K(x) = \frac{1}{2\pi} \frac{K/g}{(K/2g)^2 + x^2} , \quad (3.17)$$

$$(A_{MK} * L)(x) = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{d}{dx} \Theta_{MK}(2g(x-y)) L(y) , \quad (3.18)$$

$$\Theta_{MK}(x) = \begin{cases} \theta(\frac{x}{|K-M|}) + 2\theta(\frac{x}{|K-M|+2}) + \dots + 2\theta(\frac{x}{K+M-2}) + \theta(\frac{x}{K+M}) , & \text{if } K \neq M \\ 2\theta(\frac{x}{2}) + 2\theta(\frac{x}{4}) + \dots + 2\theta(\frac{x}{2M-2}) + \theta(\frac{x}{2M}) , & \text{if } K = M , \end{cases} \quad (3.19)$$

$$\theta(x) = 2 \arctan(x) . \quad (3.20)$$

Equations (3.14-3.20) should be compared with equations (5.43) and (5.54-5.56) in [25] evaluated at $\bar{u} \equiv u^{\text{Ref.}[25]} = 1/2g$.

In the following sections we shall derive a set of functional identities (Y-system) satisfied by the quantities $Y_A = e^{\epsilon_A}$ (or $= e^{-\epsilon_A}$). Very importantly, a Y-system is universal in the sense that it is the same for *all* the energy states $E_n(L)$, at least in a relativistic theory [29, 30]. Fugacities as those defined in (3.13) may be removed by a simple redefinition of the Y s. Therefore these are discharged in the next sections.

4 Y-system for the Hubbard model

The TBA equations for the Hubbard model in universal form are written, for example, in [25]⁵. This section is not meant to be particularly original and its aim is to explain how a subset of

⁵ See also [36] for the Y-system and the excited states in a closely-related model.

the Y-system equations proposed in [37] and in this paper emerges from the Hubbard model. The TBA equations are:

$$\begin{aligned}
\ln \eta_1(\lambda) &= s * \ln(1 + \eta_2)(\lambda) - \int_{-\pi}^{\pi} dk \cos(k) s(\lambda - \sin(k)) \ln\left(1 + \frac{1}{\zeta(k)}\right), \\
\ln \eta'_1(\lambda) &= s * \ln(1 + \eta'_2)(\lambda) - \int_{-\pi}^{\pi} dk \cos(k) s(\lambda - \sin(k)) \ln(1 + \zeta(k)), \\
\ln \eta_n(\lambda) &= s * \ln[(1 + \eta_{n-1})(1 + \eta_{n+1})](\lambda), \quad n = 2, 3, \dots, \\
\ln \eta'_n(\lambda) &= s * \ln[(1 + \eta'_{n-1})(1 + \eta'_{n+1})](\lambda), \quad n = 2, 3, \dots,
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\ln \zeta(k) &= -\frac{2}{T} \cos(k) - \frac{1}{T} \int_{-\infty}^{\infty} d\lambda s(\sin(k) - \lambda) \left(4\text{Re} \sqrt{1 - (\lambda - i\bar{u})^2}\right) \\
&+ \int_{-\infty}^{\infty} dy s(\sin(k) - \lambda) \ln\left(\frac{1 + \eta'_1}{1 + \eta_1}\right),
\end{aligned} \tag{4.2}$$

where

$$s(\lambda) = \frac{1}{4\bar{u} \cosh(\pi\lambda/2\bar{u})}, \tag{4.3}$$

is the convolution kernel. $s(\lambda)$ fulfills the following important property

$$s(\lambda + i\bar{u}) + s(\lambda - i\bar{u}) = \delta(\lambda). \tag{4.4}$$

Relation (4.4) leads to the following set of functional relations

$$\eta_n(\lambda + i\bar{u})\eta_n(\lambda - i\bar{u}) = (1 + \eta_{n-1}(\lambda))(1 + \eta_{n+1}(\lambda)), \tag{4.5}$$

$$\eta'_n(\lambda + i\bar{u})\eta'_n(\lambda - i\bar{u}) = (1 + \eta'_{n-1}(\lambda))(1 + \eta'_{n+1}(\lambda)), \tag{4.6}$$

with $n = 2, 3, \dots$. For $n = 1$ we have instead

$$\begin{aligned}
\ln[\eta_1(\lambda + i\bar{u})\eta_1(\lambda - i\bar{u})] &= \ln[(1 + \eta_2)(\lambda)] - \int_{-\pi}^{\pi} dk \cos(k) \delta(\lambda - \sin(k)) \ln\left(1 + \frac{1}{\zeta(k)}\right), \\
\ln[\eta'_1(\lambda + i\bar{u})\eta'_1(\lambda - i\bar{u})] &= \ln[(1 + \eta'_2)(\lambda)] - \int_{-\pi}^{\pi} dk \cos(k) \delta(\lambda - \sin(k)) \ln(1 + \zeta(k)).
\end{aligned}$$

But for fixed $0 < \lambda < 1$ the argument of the Dirac δ function vanishes two times, i.e. at $k = \arcsin(\lambda)$ and $k = \pi - \arcsin(\lambda)$. This gives

$$\eta_1(\lambda + i\bar{u})\eta_1(\lambda - i\bar{u}) = (1 + \eta_2(\lambda)) \left(\frac{1 + 1/\zeta(\pi - k)}{1 + 1/\zeta(k)} \right), \tag{4.7}$$

$$\eta'_1(\lambda + i\bar{u})\eta'_1(\lambda - i\bar{u}) = (1 + \eta'_2(\lambda)) \left(\frac{1 + \zeta(\pi - k)}{1 + \zeta(k)} \right). \tag{4.8}$$

Finally considering that $\cos(k) = -\sqrt{1 - \sin^2(k)}$ for $\pi/2 < k < \pi$ we get

$$\zeta^+(\pi - k)\zeta^-(\pi - k) \equiv \zeta(\pi - \arcsin(\lambda + i\bar{u}))\zeta(\pi - \arcsin(\lambda - i\bar{u})) = \left(\frac{1 + \eta'_1(\lambda)}{1 + \eta_1(\lambda)} \right). \tag{4.9}$$

From the relation

$$\zeta(\pi - k) = \zeta(k)e^{4\cos(k)/T} \quad (4.10)$$

(see eq. (5.A.2) in [25]) we also have

$$\begin{aligned} \zeta^+(k)\zeta^-(k) &\equiv \zeta(\arcsin(\lambda + i\bar{u}))\zeta(\arcsin(\lambda - i\bar{u})) = \left(\frac{1 + \eta'_1(\lambda)}{1 + \eta_1(\lambda)}\right) \\ &\times e^{\frac{4}{T}(\sqrt{1-(\sin(k)+i\bar{u})^2} + \sqrt{1-(\sin(k)-i\bar{u})^2})} . \end{aligned} \quad (4.11)$$

To see the relationship with the Y-system represented in figure 1 of [37], set $z_i = 1/\eta'_i$:

$$z_1(\lambda + i\bar{u})z_1(\lambda - i\bar{u}) = (1 + 1/z_2(\lambda))^{-1} \left(\frac{1 + \zeta(k)}{1 + \zeta(\pi - k)} \right) , \quad (4.12)$$

$$z_n(\lambda + i\bar{u})z_n(\lambda - i\bar{u}) = (1 + 1/z_{n-1}(\lambda))^{-1}(1 + 1/z_{n+1}(\lambda))^{-1} , \quad (4.13)$$

and

$$Y_{22}(k) = \zeta(k) \quad , \quad Y_{11}(k) \equiv 1/Y_{22}(\pi - k) = 1/\zeta(\pi - k) , \quad (4.14)$$

$$Y_{1,b+1}(\lambda) = z_b(\lambda) \quad , \quad Y_{a+1,1}(\lambda) = \eta_a(\lambda) , \quad (4.15)$$

with $(a, b = 1, 2, 3, \dots)$ and construct a TBA diagram using the following rules [38]:

- starting from a given node (a, b) the l.h.s of the Y-system is always $Y_{ab}(\lambda + i\bar{u})Y_{ab}(\lambda - i\bar{u})$;
- an horizontal link between the node (a, b) and (a', b) corresponds to a factor $(1 + Y_{a'b}(\lambda))$ on the r.h.s. ;
- a vertical link between (a, b) and (a, b') corresponds to a factor $(1 + 1/Y_{ab'}(\lambda))^{-1}$ on the r.h.s. .

It is easy to check that the diagram represented in figure 1 is reproduced with the exception of the functional relation (4.11) for $Y_{22}(\lambda(k)) = \zeta(k)$ which would close a ‘standard’ Y-system diagram only if this extra constraint were true

$$\frac{\eta_1(\lambda(k))}{\eta'_1(\lambda(k))} = e^{\frac{4}{T}(\sqrt{1-(\sin(k)+i\bar{u})^2} + \sqrt{1-(\sin(k)-i\bar{u})^2})} . \quad (4.16)$$

This equation certainly holds at $T = \infty$ and would be compatible with some of the evident symmetries of the TBA equations but still it would imply a chain of extra constraints (on the other TBA functions) that we did not try to prove. In fact, we should stress that we have included the node Y_{11} , which is related to Y_{22} by (4.14) and (4.10). Therefore, there is no need to show an extra equation for $Y_{22}(\lambda(k)) = \zeta(k)$ ⁶, once we already have $\ln Y_{11}(\lambda(k)) = -\ln \zeta(\pi - k)$ in the TBA system.

⁶A *fortiori*, if this equation should not respect the ‘standard’ form of the Y-system.

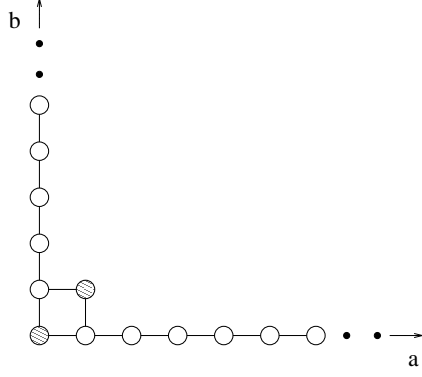


Figure 1: The Hubbard diagram

5 Y-system for the AdS/CFT correspondence

Let us start from equation (3.8) and observe that the $S_{KM}(u)$ defined in (2.13) are a particular $n \rightarrow \infty$ limit of the Z_n -related scattering matrix elements proposed in [39]. They satisfy the following set of functional relations [27, 40]

$$S_{KM} \left(2\lambda + \frac{i}{g} \right) S_{KM} \left(2\lambda - \frac{i}{g} \right) = \prod_{K'=1}^{\infty} (S_{K'M}(2\lambda))^{I_{KK'}} e^{-i2\pi I_{KM} \Theta(2\lambda)} , \quad (5.1)$$

where $I_{NM} = \delta_{N,M+1} + \delta_{N,M-1}$ and $\Theta(u)$ is the Heaviside step function. Equation (5.1) leads to

$$\phi_{ww}^{KM} \left(\lambda' - \lambda + \frac{i}{2g} \right) + \phi_{ww}^{KM} \left(\lambda' - \lambda - \frac{i}{2g} \right) - \sum_{K'=1}^{\infty} I_{KK'} \phi_{ww}^{K'M}(\lambda' - \lambda) = -I_{KM} \delta(\lambda' - \lambda) . \quad (5.2)$$

Notice that $\phi_{ww}^{KM}(\lambda)$ is equal to $-A_{KM}(\lambda)$ defined in equation (3.18). Another relevant identity is

$$\begin{aligned} & \phi_{yw}^K \left(\sin(q'), \lambda + \frac{i}{2g} \right) + \phi_{yw}^K \left(\sin(q'), \lambda - \frac{i}{2g} \right) - \\ & \sum_{K'=1}^{\infty} I_{KK'} \phi_{yw}^{K'}(\sin(q'), \lambda) = -\delta_{K1} \cos(q') \delta(\sin(q') - \lambda) . \end{aligned} \quad (5.3)$$

Using equations (5.2), (5.3) and setting

$$Y_{w,K}^{\alpha}(\lambda) = e^{-\epsilon_{w,K}^{\alpha}(\lambda)} , \quad Y_y^{\alpha}(q) = e^{\epsilon_y^{\alpha}(q)} , \quad Y_{y^*}^{\alpha}(q) \equiv e^{\epsilon_{y^*}^{\alpha}(q)} = e^{-\epsilon_y^{\alpha}(\pi-q)} , \quad (5.4)$$

with $q = \arcsin(\lambda)$, we find

$$Y_{w,K}^{\alpha}(\lambda + \frac{i}{2g}) Y_{w,K}^{\alpha}(\lambda - \frac{i}{2g}) = \prod_{K'=1}^{\infty} \left(1 + \frac{1}{Y_{w,K'}^{\alpha}(\lambda)} \right)^{-I_{KK'}} \left(\frac{1 + Y_{y^*}^{\alpha}(q)}{1 + 1/Y_y^{\alpha}(q)} \right)^{\delta_{K1}} . \quad (5.5)$$

Let us now consider equation (3.7). The identity (8.8) with $K = 2, 3, \dots$, together with equations (5.2) and (5.3) lead to

$$Y_{v,K}^\alpha(\lambda + \frac{i}{2g})Y_{v,K}^\alpha(\lambda - \frac{i}{2g}) = \prod_{K'=1}^{\infty} (1 + Y_{v,K'}^\alpha(\lambda))^{I_{KK'}} \left(1 + \frac{1}{Y_{K+1}(\tilde{p})}\right)^{-1}, \quad (5.6)$$

with $\tilde{p} = \tilde{p}(2\lambda)$ defined in (7.10) and $Y_{v,K}^\alpha = e^{\epsilon_{v,K}^\alpha}$. The case with $K = 1$ is slightly more tricky, but the game is just the same. One starts considering the expression

$$\epsilon_{v1}^\alpha(\lambda + \frac{i}{2g}) + \epsilon_{v1}^\alpha(\lambda - \frac{i}{2g}) - \epsilon_{v2}^\alpha(\lambda) - \epsilon_y^\alpha(q) - \epsilon_{y^*}^\alpha(q), \quad (5.7)$$

with $q = \arcsin(\lambda)$. The corresponding r.h.s. of the TBA equations cancel almost completely due to the functional relations fulfilled by the kernel functions, they just leave some ‘contact’ delta function contributions. In this case the result is

$$\begin{aligned} Y_{v,1}^\alpha(\lambda + \frac{i}{2g})Y_{v,1}^\alpha(\lambda - \frac{i}{2g}) &= (1 + Y_{v,2}^\alpha(\lambda))(1 + Y_y^\alpha(q)) \\ &\times \left(1 + \frac{1}{Y_{y^*}(q)}\right)^{-1} \left(1 + \frac{1}{Y_2(\tilde{p})}\right)^{-1}. \end{aligned} \quad (5.8)$$

Further, consider the quantity

$$\epsilon_y(q^+) + \epsilon_y(q^-) - \epsilon_{v,1}(\lambda), \quad (5.9)$$

where $q^\pm = \arcsin(\lambda \pm i/2g)$, the kernel properties and the TBA equation (3.8) at $K = 1$ give

$$Y_y^\alpha(q^+)Y_y^\alpha(q^-) = (1 + Y_{v,1}^\alpha(\lambda)) \left(1 + \frac{1}{Y_{w,1}^\alpha(\lambda)}\right)^{-1} \left(1 + \frac{1}{Y_1(\tilde{p})}\right)^{-1}, \quad (5.10)$$

with $\tilde{p} = \tilde{p}(2\lambda)$. Finally, using the property

$$\frac{x^{Q^+}(u + i/g) x^{Q^+}(u - i/g)}{x^{Q^-}(u + i/g) x^{Q^-}(u - i/g)} = \frac{x^{(Q-1)^+}(u) x^{(Q+1)^+}(u)}{x^{(Q-1)^-}(u) x^{(Q+1)^-}(u)}, \quad (5.11)$$

and similar relations for $\phi_{\text{sl}(2)}^{Q'Q}$, ϕ_{yx}^Q and ϕ_{vx}^{QM} (see Appendix B) we get

$$Y_Q(x(u + \frac{i}{g}))Y_Q(x(u - \frac{i}{g})) = \prod_{Q'=1}^{\infty} (1 + Y_{Q'}(x(u)))^{I_{QQ'}} \prod_{\alpha=1}^2 \left(1 + \frac{1}{Y_{v,Q-1}^\alpha(\lambda)}\right)^{-1}, \quad (5.12)$$

with $Q = 2, 3, \dots$ and

$$Y_1(x(u + \frac{i}{g}))Y_1(x(u - \frac{i}{g})) = (1 + Y_2(x(u))) \prod_{\alpha=1}^2 \left(1 + \frac{1}{Y_y^\alpha(q)}\right)^{-1}, \quad (5.13)$$

with $q = \arcsin(\lambda)$, $u = 2\lambda$ and $Y_Q = e^{\epsilon_Q}$. Setting

$$\begin{aligned} Y_{Q,0} &= Y_Q \quad , \quad Y_{1,1} = Y_y^1 \quad , \quad Y_{1,-1} = Y_y^2 \quad , \quad Y_{2,2} = Y_{y^*}^1 \quad , \quad Y_{2,-2} = Y_{y^*}^2 \quad , \\ Y_{1,K+1} &= Y_{w,K}^1 \quad , \quad Y_{1,-K-1} = Y_{w,K}^2 \quad , \quad Y_{K+1,1} = Y_{v,K}^1 \quad , \quad Y_{K+1,-1} = Y_{v,K}^2 \quad , \end{aligned} \quad (5.14)$$

and following the rules given at the end of section 4 we may encode this Y-system in the diagram in figure 2. In other words, the equations (5.5-5.13) with the identifications (5.14) can be recast in the compact form

$$Y_{a,b}^+ Y_{a,b}^- = (1 + Y_{a+1,b})(1 + Y_{a-1,b}) \left(1 + \frac{1}{Y_{a,b+1}}\right)^{-1} \left(1 + \frac{1}{Y_{a,b-1}}\right)^{-1} \quad , \quad (5.15)$$

as long as $(a, b) \neq (2, \pm 2)$.

Our Y-diagram shares its structure with that in figure 1 of [37]. Yet, we shall remark the exact parallel to what we have noticed at the end of section 4 about the Hubbard model: to close completely the diagram by using the ‘standard’ rules, we would need two extra equations

$$Y_{y^*}^\alpha(q^+) Y_{y^*}^\alpha(q^-) = (1 + Y_{w,1}^\alpha(\lambda)) \left(1 + \frac{1}{Y_{v,1}^\alpha(\lambda)}\right)^{-1} \quad , \quad (\alpha = 1, 2) \quad . \quad (5.16)$$

The careful reader may have noticed that we did not prove these equations, since for the nodes $(2, \pm 2)$ we already have the identification

$$Y_{2,\pm 2}^\alpha(q) = \frac{1}{Y_{1,\pm 1}^\alpha(\pi - q)} \quad , \quad (5.17)$$

and thus, at any rate, we do not need to include the associated equations in the TBA system. A careful analysis suggests that equation (5.16) is in general incorrect.

6 Partial conclusions and remarks

In a nutshell, we have proposed the TBA equations which should control the energy/dimension spectrum of the $\text{AdS}_5 \times \text{S}^5$ correspondence. We have also derived from them the universal Y-system which should characterise any state of the theory for any value of the coupling constant g . Of course, since universal, this system contains the information about a specific state in a much more involved way.

Nevertheless, we may still lean on the theory of massive integrable field theories. In this area a clear procedure has been established to extract excited state non-linear integral equations from that of the ground state: this is initially described from three different perspectives in the papers [29, 30, 31]. Essentially, it proves the recipe to extract suitable driving terms $\sum_i \ln S(u_i, u)$ as residues of the convolution integrals, and these terms clearly involve the scattering matrix elements. Under the perspective of the non-linear integral equation, this idea has been already applied to some sectors of the asymptotic Beisert-Staudacher equations [41, 42, 43].

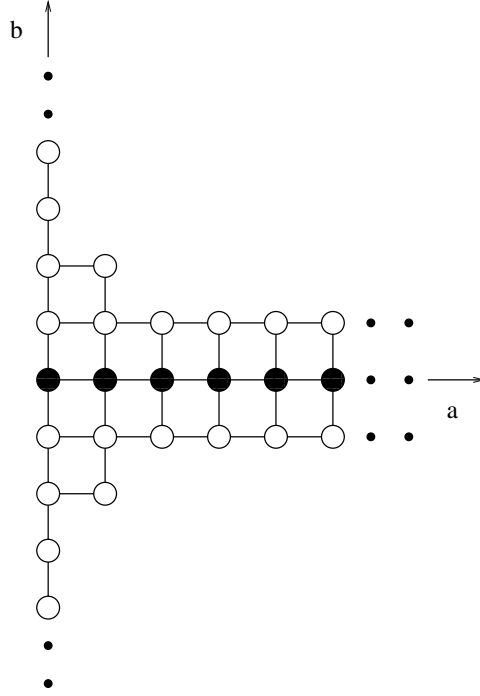


Figure 2: The AdS/CFT diagram

Re-echoing the title of [44], the Hubbard model excursion seems to be still on in this discipline. In fact, we have found just two copies of this model, talking through their massive nodes. Moreover, this is also the structure of the Y -system recently proposed by Gromov, Kazakov and Viera (somehow on symmetry grounds) [37].

Despite the lack of a BA or integrability description of sufficiently *short* operators, we may consider all these arguments in favour of a TBA description of the correspondence.

Acknowledgements

DF is particularly indebted to M. Rossi for insightful discussions and suggestions. We also thank G. Arutyunov and F. Ravanini. We acknowledge the INFN grants IS PI14 “*Topics in non-perturbative gauge dynamics in field and string theory*” and PI11 for travel financial support, and the University PRIN 2007JHLPEZ “Fisica Statistica dei Sistemi Fortemente Correlati all’Equilibrio e Fuori Equilibrio: Risultati Esatti e Metodi di Teoria dei Campi”.

7 Appendix A

Here we report the definitions used for the kernels involved in the TBA equations (3.5)-(3.8):

$$\phi_{\text{sl}(2)}^{Q'Q}(\tilde{p}', \tilde{p}) = \frac{1}{2\pi i} \frac{d}{d\tilde{p}'} \ln S_{\text{sl}(2)}^{Q'Q}(\tilde{p}', \tilde{p}) , \quad (7.1)$$

$$\phi_{xy}^Q(\tilde{p}, q) = \frac{1}{2\pi i} \frac{d}{d\tilde{p}} \ln \left(\frac{x^{Q-}(\tilde{p}) - y(q)}{x^{Q+}(\tilde{p}) - y(q)} \sqrt{\frac{x^{Q+}(\tilde{p})}{x^{Q-}(\tilde{p})}} \right) , \quad (7.2)$$

$$\phi_{xv}^{QM}(\tilde{p}, \lambda) = \frac{1}{2\pi i} \frac{d}{d\tilde{p}} \ln S_{xv}^{QM}(\tilde{p}, \lambda) , \quad (7.3)$$

$$\phi_{vx}^{KQ}(\lambda, \tilde{p}) = -\frac{1}{2\pi i} \frac{d}{d\lambda} \ln S_{xv}^{QK}(\tilde{p}, \lambda) , \quad (7.4)$$

$$\phi_{yx}^Q(q, \tilde{p}) = \frac{1}{2\pi i} \frac{d}{dq} \ln \left(\frac{y(q) - x^{Q-}(\tilde{p})}{y(q) - x^{Q+}(\tilde{p})} \sqrt{\frac{x^{Q+}(\tilde{p})}{x^{Q-}(\tilde{p})}} \right) , \quad (7.5)$$

$$\phi_{yv}^K(q, \lambda) = \phi_{yw}^K(q, \lambda) = \frac{1}{2\pi i} \frac{d}{dq} \ln \left(\frac{v(q) - 2\lambda - iK/g}{v(q) - 2\lambda + iK/g} \right) , \quad (7.6)$$

$$\phi_{vv}^{MK}(\lambda', \lambda) = \phi_{ww}^{MK}(\lambda', \lambda) = \frac{1}{2\pi i} \frac{d}{d\lambda'} \ln S_{MK}(2\lambda' - 2\lambda) , \quad (7.7)$$

$$\phi_{vy}^K(\lambda, q) = -\phi_{wy}^K(\lambda, q) = \frac{1}{2\pi i} \frac{d}{d\lambda} \ln \left(\frac{2\lambda - v(q) + iK/g}{2\lambda - v(q) - iK/g} \right) , \quad (7.8)$$

where

$$x^{Q\pm}(\tilde{p}) = \frac{1}{2g} \left(\sqrt{1 + \frac{4g^2}{Q^2 + \tilde{p}^2}} \mp 1 \right) (\tilde{p} - iQ) , \quad (7.9)$$

$$\tilde{p}(u) = \frac{ig}{2} \left(\sqrt{4 - \left(u + i\frac{Q}{g}\right)^2} - \sqrt{4 - \left(u - i\frac{Q}{g}\right)^2} \right) , \quad (7.10)$$

$$y(q) = i e^{-iq} , \quad v(q) = 2 \sin(q) , \quad w(\lambda) = 2\lambda , \quad (7.11)$$

$$v_K^\pm(\lambda) = 2\lambda_{v,K} \pm \frac{iK}{g} , \quad w_K^\pm(\lambda) = 2\lambda_{w,K} \pm \frac{iK}{g} , \quad (7.12)$$

$$x(u) = \frac{1}{2} \left(u - i\sqrt{4 - u^2} \right) , \quad x^{Q\pm}(-u) = -\frac{1}{x^{Q\mp}(u)} , \quad (7.13)$$

$$x^{Q\pm}(u) = x(u \pm i\frac{Q}{g}) , \quad \tilde{p}(-u) = -\tilde{p}(u) . \quad (7.14)$$

It is easy to notice that some of these kernels depends only on the difference of the rapidities, as in the relativistic case. They are

$$\phi_{vy}^M(\lambda, q) = -\phi_{wy}^M(\lambda, q) = \phi_M(\lambda - \sin(q)) , \quad \text{where } \phi_M(\lambda) = \frac{1}{2\pi i} \frac{d}{d\lambda} \ln \left(\frac{\lambda + iM/2g}{\lambda - iM/2g} \right) \quad (7.15)$$

$$\phi_{vv}^{MK}(\lambda', \lambda) = \phi_{ww}^{MK}(\lambda', \lambda) = \phi_{MK}(\lambda' - \lambda) , \quad \text{where } \phi_{MK}(\lambda) = \frac{1}{2\pi i} \frac{d}{d\lambda} \ln S_{MK}(2\lambda) . \quad (7.16)$$

8 Appendix B

Here we want to show how also the other kernels satisfy an identity of the type (5.2). As long as the kernel

$$\phi_{\mathfrak{sl}(2)}^{QQ'}(u, u') = \frac{1}{2\pi i} \frac{d}{d\tilde{p}} \ln \left[\left(\frac{u - u' + i \frac{|Q-Q'|}{g}}{u - u' - i \frac{|Q-Q'|}{g}} \right) \left(\frac{u - u' + i \frac{Q+Q'}{g}}{u - u' - i \frac{Q+Q'}{g}} \right) \right. \\ \left. \left[\frac{1 - \frac{1}{x_k^{Q+} x_l^{Q'-}}}{1 - \frac{1}{x_k^{Q-} x_l^{Q'+}}} \sigma(x_k^{Q\pm}, x_l^{Q'\pm}) \right]^{-2} \prod_{k=1}^{\min(Q, Q')-1} \left(\frac{u - u' + i \frac{|Q-Q'|+2k}{g}}{u - u' - i \frac{|Q-Q'|+2k}{g}} \right)^2 \right] \quad (8.1)$$

is concerned, we may shift on the second variable

$$\phi_{\mathfrak{sl}(2)}^{QQ'} \left(u, u' + \frac{i}{g} \right) + \phi_{\mathfrak{sl}(2)}^{QQ'} \left(u, u' - \frac{i}{g} \right) = \frac{1}{2\pi i} \frac{d}{d\tilde{p}} \left[\ln \left(\frac{u - u' + i \frac{|Q-Q'|}{g} - \frac{i}{g}}{u - u' - i \frac{|Q-Q'|}{g} - \frac{i}{g}} \right) \right. \\ + \ln \left(\frac{u - u' + i \frac{Q+Q'-1}{g}}{u - u' - i \frac{Q+Q'+1}{g}} \right) + 2 \sum_{k=1}^{\min(Q, Q'-1)-1} \ln \left(\frac{u - u' + i \frac{|Q-Q'|+2k}{g} - \frac{i}{g}}{u - u' - i \frac{|Q-Q'|+2k}{g} - \frac{i}{g}} \right) \\ + \ln \left(\frac{u - u' + i \frac{|Q-Q'|}{g} + \frac{i}{g}}{u - u' - i \frac{|Q-Q'|}{g} + \frac{i}{g}} \right) + \ln \left(\frac{u - u' + i \frac{Q+Q'+1}{g}}{u - u' - i \frac{Q+Q'-1}{g}} \right) \\ + 2 \sum_{k=1}^{\min(Q, Q'+1)-1} \ln \left(\frac{u - u' + i \frac{|Q-Q'|+2k}{g} + \frac{i}{g}}{u - u' - i \frac{|Q-Q'|+2k}{g} + \frac{i}{g}} \right) - 2 \ln \left(\frac{1 - \frac{1}{x(u - \frac{iQ}{g}) x(u + i \frac{Q'+1}{g})}}{1 - \frac{1}{x(u + \frac{iQ}{g}) x(u - i \frac{Q'-1}{g})}} \right) \\ - 2 \ln \left(\frac{1 - \frac{1}{x(u - \frac{iQ}{g}) x(u + i \frac{Q'-1}{g})}}{1 - \frac{1}{x(u + \frac{iQ}{g}) x(u - i \frac{Q'+1}{g})}} \right) \\ - 2i \sum_{r=2}^{\infty} \sum_{\nu=0}^{\infty} \beta_{r, r+1+2\nu}(g) [q_r^Q(u) q_{r+1+2\nu}^{Q'+1}(u') - q_r^{Q'+1}(u') q_{r+1+2\nu}^Q(u) \\ + q_r^Q(u) q_{r+1+2\nu}^{Q'-1}(u') - q_r^{Q'-1}(u') q_{r+1+2\nu}^Q(u)] \Bigg] , \quad (8.2)$$

so proving the identity used in the main text

$$\phi_{\mathfrak{sl}(2)}^{QQ'} \left(u, u' + \frac{i}{g} \right) + \phi_{\mathfrak{sl}(2)}^{QQ'} \left(u, u' - \frac{i}{g} \right) = \sum_{Q''=1}^{\infty} I_{Q'Q''} \phi_{\mathfrak{sl}(2)}^{QQ''}(u, u') - \delta(u - u') I_{QQ'} . \quad (8.3)$$

The bound state charges q_r^Q are defined as usual [45, 46] and the shifted charges we use above are

$$q_r^{Q\pm 1}(u) = \frac{i}{r-1} \left[\left(\frac{1}{x \left(u + \frac{i(Q\pm 1)}{g} \right)} \right)^{r-1} - \left(\frac{1}{x \left(u - \frac{i(Q\mp 1)}{g} \right)} \right)^{r-1} \right]. \quad (8.4)$$

Analogously, by direct computation

$$\begin{aligned} \phi_{vx}^{MQ} \left(\lambda, x^{Q\pm} \left(u + \frac{i}{g} \right) \right) + \phi_{vx}^{MQ} \left(\lambda, x^{Q\pm} \left(u - \frac{i}{g} \right) \right) &= -\frac{1}{2\pi i} \frac{d}{d\lambda} \left[\ln \left(\frac{x^{(Q-1)-} - x \left(v + \frac{iM}{g} \right)}{x^{(Q+1)+} - x \left(v + \frac{iM}{g} \right)} \right) \right. \\ &+ \ln \left(\frac{x^{(Q+1)-} - x \left(v - \frac{iM}{g} \right)}{x^{(Q-1)+} - x \left(v - \frac{iM}{g} \right)} \right) + \ln \left(\frac{x^{(Q+1)+}}{x^{(Q-1)-}} \right) \\ &+ \ln \left(\frac{x^{(Q-1)+}}{x^{(Q+1)-}} \right) + \ln \left(\frac{x^{(Q+1)-} - x \left(v + \frac{iM}{g} \right)}{x^{(Q-1)+} - x \left(v + \frac{iM}{g} \right)} \right) + \sum_{j=1}^{M-1} \left[\ln \left(\frac{u - i \frac{Q-1}{g} - \left(v - \frac{iM}{g} \right) - \frac{2i}{g} j}{u + i \frac{Q+1}{g} - \left(v + \frac{iM}{g} \right) + \frac{2i}{g} j} \right) \right. \\ &\left. \left. + \ln \left(\frac{u - i \frac{Q+1}{g} - \left(v - \frac{iM}{g} \right) - \frac{2i}{g} j}{u + i \frac{Q-1}{g} - \left(v + \frac{iM}{g} \right) + \frac{2i}{g} j} \right) \right] \right], \quad (8.5) \end{aligned}$$

($v = 2\lambda$), we may prove an identity with the same form, but involving ϕ_{vx}^{MQ}

$$\begin{aligned} \phi_{vx}^{MQ} \left(\lambda, x^{Q\pm} \left(u + \frac{i}{g} \right) \right) + \phi_{vx}^{MQ} \left(\lambda, x^{Q\pm} \left(u - \frac{i}{g} \right) \right) &= \sum_{Q'=1}^{\infty} I_{QQ'} \phi_{vx}^{MQ'} \left(\lambda, x^{Q'}(u) \right) \\ &+ \delta(\lambda - u/2) \delta_{Q-1, M}. \quad (8.6) \end{aligned}$$

An identity with the same form may be derived for ϕ_{xv}^{QM} :

$$\begin{aligned} \phi_{xv}^{QM} \left(x^{Q\pm}(u), \lambda + \frac{i}{2g} \right) + \phi_{xv}^{QM} \left(x^{Q\pm}(u), \lambda - \frac{i}{2g} \right) &= \frac{1}{2\pi i} \frac{d}{d\tilde{p}} \left[\ln \left(\frac{x^{Q-} - x \left(v + \frac{i(M+1)}{g} \right)}{x^{Q+} - x \left(v + \frac{i(M+1)}{g} \right)} \right) \right. \\ &+ \ln \left(\frac{x^{Q-} - x \left(v - \frac{i(M-1)}{g} \right)}{x^{Q+} - x \left(v - \frac{i(M-1)}{g} \right)} \right) + \ln \left(\frac{x^{Q-} - x \left(v - \frac{i(M+1)}{g} \right)}{x^{Q+} - x \left(v - \frac{i(M+1)}{g} \right)} \right) + 2 \ln \left(\frac{x^{Q+}}{x^{Q-}} \right) \\ &+ \ln \left(\frac{x^{Q-} - x \left(v + \frac{i(M-1)}{g} \right)}{x^{Q+} - x \left(v + \frac{i(M-1)}{g} \right)} \right) + \sum_{j=1}^{M-1} \left[\ln \left(\frac{u - i \frac{Q}{g} - \left(v - i \frac{M-1}{g} \right) - \frac{2i}{g} j}{u + i \frac{Q}{g} - \left(v + i \frac{M+1}{g} \right) + \frac{2i}{g} j} \right) \right. \\ &\left. \left. + \ln \left(\frac{u - i \frac{Q}{g} - \left(v - i \frac{M+1}{g} \right) - \frac{2i}{g} j}{u + i \frac{Q}{g} - \left(v + i \frac{M-1}{g} \right) + \frac{2i}{g} j} \right) \right] \right], \quad (8.7) \end{aligned}$$

$$\begin{aligned} \phi_{xv}^{QM} \left(x^{Q\pm}(u), \lambda + \frac{i}{2g} \right) + \phi_{xv}^{QM} \left(x^{Q\pm}(u), \lambda - \frac{i}{2g} \right) &= \sum_{M'=1}^{\infty} I_{MM'} \phi_{xv}^{QM'} \left(x^Q(u), \lambda \right) \\ &+ \delta(\lambda - u/2) \delta_{Q-1, M}. \quad (8.8) \end{aligned}$$

References

- [1] H. Bethe, “On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain”, *Z. Phys.* **71** (1931) 205;
C. N. Yang and C. P. Yang, “One-dimensional chain of anisotropic spin-spin interactions. I: Proof of Bethe’s hypothesis for ground state in a finite system”, *Phys. Rev.* **150** (1966) 321;
C. N. Yang, “Some exact results for the many body problems in one dimension with repulsive delta function interaction”, *Phys. Rev. Lett.* **19** (1967) 1312;
R.J. Baxter, “Partition function of the eight-vertex model”, *Ann. Phys.* **70** (1972) 193;
L. D. Faddeev, E. K. Sklyanin and L. A. Takhtajan, “The Quantum Inverse Problem Method. 1”, *Theor. Math. Phys.* **40** (1980) 688 [*Teor. Mat. Fiz.* **40** (1979) 194];
A. B. Zamolodchikov and A. B. Zamolodchikov, “Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models”, *Annals Phys.* **120** (1979) 253.
- [2] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity”, *Adv. Theor. Math. Phys.* **2** (1998) 231, [arXiv:hep-th/9711200];
S.S. Gubser, I.R. Klebanov, A.M. Polyakov, “Gauge theory correlators from non-critical string theory”, *Phys.Lett.* **B428** (1998) 105, [arXiv:hep-th/9802109];
E. Witten, “Anti-de Sitter space and holography”, *Adv. Theor. Math. Phys.* **2** (1998) 253, [arXiv:hep-th/9802150].
- [3] L. N. Lipatov, “High-energy asymptotics of multicolor QCD and exactly solvable lattice models”, [arXiv:hep-th/9311037];
L. D. Faddeev and G. P. Korchemsky, “High-energy QCD as a completely integrable model”, *Phys. Lett. B* **342** (1995) 311 [arXiv:hep-th/9404173].
- [4] I. Bena, J. Polchinski, R. Roiban, “Hidden symmetries of the $AdS_5 \times S^5$ superstring”, *Phys. Rev. D* **69** (2004) 046002, [arXiv:hep-th/0305116].
- [5] L.N. Lipatov, ‘Evolution equations in QCD’, in “Perspectives in Hadron Physics”, Proceedings of the Conference, ICTP, Trieste, Italy, May 1997, World Scientific (Singapore, 1998).
- [6] J.A. Minahan, K. Zarembo, “The Bethe Ansatz for $\mathcal{N} = 4$ Super Yang-Mills”, *JHEP***03** (2003) 013, [arXiv:hep-th/0212208].
- [7] N. Beisert, M. Staudacher, “Long-range $psu(2,2|4)$ Bethe Ansätze for gauge theory and strings”, *Nucl. Phys. B* **727** (2005) 1 [arXiv:hep-th/0504190].
- [8] G. Arutyunov, S. Frolov and M. Staudacher, “Bethe ansatz for quantum strings”, *JHEP* **0410** (2004) 016 [arXiv:hep-th/0406256].
- [9] R. Hernandez and E. Lopez, “Quantum corrections to the string Bethe ansatz”, *JHEP* **0607** (2006) 004 [arXiv:hep-th/0603204].
- [10] N. Beisert, R. Hernandez and E. Lopez, “A crossing-symmetric phase for $AdS(5) \times S^{*5}$ strings”, *JHEP* **0611** (2006) 070 [arXiv:hep-th/0609044].

- [11] N. Beisert, B. Eden and M. Staudacher, “Transcendentality and crossing”, J. Stat. Mech. **0701** (2007) P021 [arXiv:hep-th/0610251].
- [12] C. Sieg and A. Torrielli, Nucl. Phys. B **723**, 3 (2005) [arXiv:hep-th/0505071].
- [13] J. Ambjorn, R. A. Janik and C. Kristjansen, “Wrapping interactions and a new source of corrections to the spin-chain / string duality”, Nucl. Phys. B **736** (2006) 288 [arXiv:hep-th/0510171];
- [14] M. Staudacher, “The factorized S-matrix of CFT/AdS,” JHEP **0505** (2005) 054 [arXiv:hep-th/0412188].
- [15] N. Beisert, “The $su(2|2)$ dynamic S-matrix”, Adv. Theor. Math. Phys. **12** (2008) 945 [arXiv:hep-th/0511082]; M. J. Martins and C. S. Melo, “The Bethe ansatz approach for factorizable centrally extended S-matrices”, Nucl. Phys. B **785** (2007) 246 [arXiv:hep-th/0703086].
- [16] N. Beisert, “The Analytic Bethe Ansatz for a Chain with Centrally Extended $su(2|2)$ Symmetry”, J. Stat. Mech. **0701** (2007) P017 [arXiv:nlin/0610017];
- [17] M. Luscher, “Volume Dependence Of The Energy Spectrum In Massive Quantum Field Theories. 1. Stable Particle States”, Commun. Math. Phys. **104** (1986) 177; “On A Relation Between Finite Size Effects And Elastic Scattering Processes”, Lecture given at Cargese Summer Inst., Cargese, France, Sep 1-15, 1983.
- [18] T. R. Klassen and E. Melzer, “On the relation between scattering amplitudes and finite size mass corrections in QFT”, Nucl. Phys. B **362** (1991) 329.
- [19] Z. Bajnok and R. A. Janik, “Four-loop perturbative Konishi from strings and finite size effects for multiparticle states”, Nucl. Phys. B **807** (2009) 625 [arXiv:0807.0399 [hep-th]]; Z. Bajnok, R. A. Janik and T. Lukowski, “Four loop twist two, BFKL, wrapping and strings”, arXiv:0811.4448 [hep-th].
- [20] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, “Wrapping at four loops in $N=4$ SYM”, Phys. Lett. B **666** (2008) 100 [arXiv:0712.3522 [hep-th]]; “Anomalous dimension with wrapping at four loops in $N=4$ SYM”, Nucl. Phys. B **805** (2008) 231 [arXiv:0806.2095 [hep-th]].
- [21] G. Arutyunov and S. Frolov, “String hypothesis for the $AdS_5 \times S^5$ mirror”, arXiv:0901.1417 [hep-th].
- [22] G. Arutyunov and S. Frolov, “On String S-matrix, Bound States and TBA”, JHEP **0712** (2007) 024 [arXiv:0710.1568 [hep-th]].
- [23] C.N. Yang and C. F. Yang, “Thermodynamics of one-dimensional system of bosons with repulsive delta function interaction”, J. Math. Phys. **10** (1969) 1115;
- [24] M. Takahashi, “One-Dimensional Hubbard Model at Finite Temperature”, Prog. Theor. Phys., **47** (1972) 69.
- [25] F.H.L. Essler, H.Frahm, F.Gohmann, A.Klumper and V.E.Korepin, “*The one-dimensional Hubbard Model*”, Cambridge University press.

- [26] A.I.B. Zamolodchikov, “Thermodynamic Bethe ansatz in relativistic models. Scaling three state Potts and Lee-Yang models”, Nucl. Phys. B **342**, 695 (1990).
- [27] A. B. Zamolodchikov, “On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories”, Phys. Lett. B **253** (1991) 391.
- [28] T. R. Klassen and E. Melzer, “The thermodynamics of purely elastic scattering theories and conformal perturbation theory”, Nucl. Phys. B **350** (1991) 635.
- [29] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, “Quantum field theories in finite volume: Excited state energies”, Nucl. Phys. B **489**, 487 (1997) [arXiv:hep-th/9607099].
- [30] P. Dorey and R. Tateo, “Excited states by analytic continuation of TBA equations”, Nucl. Phys. B **482** (1996) 639 [arXiv:hep-th/9607167].
- [31] D. Fioravanti, A. Mariottini, E. Quattrini, F. Ravanini, “Excited state Destri-de Vega equation for sine-Gordon and restricted sine-Gordon models”, Phys. Lett. **B390** (1997) 243 and hep-th/9608091;
- [32] P. Fendley, “Excited state thermodynamics”, Nucl. Phys. B **374** (1992) 667 [arXiv:hep-th/9109021].
- [33] M. J. Martins, “Complex excitations in the thermodynamic Bethe ansatz approach”, Phys. Rev. Lett. **67** (1991) 419.
- [34] P. Fendley and K. A. Intriligator, “Scattering and thermodynamics of fractionally charged supersymmetric solitons”, Nucl. Phys. B **372** (1992) 533 [arXiv:hep-th/9111014].
- [35] P. Dorey, A. Pocklington and R. Tateo, “Integrable aspects of the scaling q-state Potts models. II: Finite-size effects”, Nucl. Phys. B **661** (2003) 464 [arXiv:hep-th/0208202].
- [36] G. Juttner, A. Klumper and J. Suzuki, “From fusion hierarchy to excited state TBA”, Nucl. Phys. B **512** (1998) 581 [arXiv:hep-th/9707074].
- [37] N. Gromov, V. Kazakov and P. Vieira, “Integrability for the Full Spectrum of Planar AdS/CFT”, [arXiv:0901.3753 [hep-th]].
- [38] E. Quattrini, F. Ravanini and R. Tateo, ‘Integrable QFT in two-dimensions encoded on products of Dynkin diagrams’, [arXiv:hep-th/9311116].
- [39] R. Koberle and J. A. Swieca, “Factorizable Z(N) Models”, Phys. Lett. B **86**, 209 (1979).
- [40] F. Ravanini, R. Tateo and A. Valleriani, “Dynkin TBAs”, Int. J. Mod. Phys. A **8** (1993) 1707 [arXiv:hep-th/9207040].
- [41] D. Fioravanti and M. Rossi, “On the commuting charges for the highest dimension SU(2) operator in planar $\mathcal{N} = 4$ SYM”, JHEP **0708** (2007) 089 [arXiv:0706.3936 [hep-th]].
- [42] L. Freyhult, A. Rej and M. Staudacher, “A Generalized Scaling Function for AdS/CFT”, J. Stat. Mech. **0807** (2008) P07015 [arXiv:0712.2743 [hep-th]].
- [43] D. Bombardelli, D. Fioravanti and M. Rossi, “Large spin corrections in $\mathcal{N} = 4$ SYM sl(2): still a linear integral equation”, Nucl. Phys. B **810** (2009) 460 [arXiv:0802.0027 [hep-th]].

- [44] G. Feverati, D. Fioravanti, P. Grinza and M. Rossi, “Hubbard’s adventures in $N = 4$ SYM-land? Some non-perturbative considerations on finite length operators”, J. Stat. Mech. **0702** (2007) P001 [arXiv:hep-th/0611186].
- [45] R. Roiban, “Magnon bound-state scattering in gauge and string theory”, JHEP **0704** (2007) 048 [arXiv:hep-th/0608049].
- [46] H. Y. Chen, N. Dorey and K. Okamura, “On the scattering of magnon boundstates”, JHEP **0611** (2006) 035 [arXiv:hep-th/0608047].